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Transformation Laws for Decomposable World-Spin Affinities in a Class of Conformally Flat Spacetimes

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Abstract A class of conformally flat spacetimes that admit certain decomposable world-spin affine patterns is considered. New coordinate-derivative relations are particularly utilized to demonstrate that the procedures involved in a well-known spinor translation of the corresponding Riemann and Ricci tensors bear world invariance. The establishment of this invariance property will presumably shed some light upon the overall spacetime situation taken up by the underlying works.

Keywords Decomposable world-spin affinities · Conformally flat spacetimes · Transformation laws

1 Introduction

A decomposable Christoffel connexion associated to a particular spin-affine model was utilized many years ago by Penrose and Rindler [1, 2] for describing some properties of a class of conformally flat spacetimes. The geometric significance of the affine structures taken up explicitly in this work was sorted out much later in conjunction with a spinor translation of the corresponding Riemann tensor and a systematic derivation of some cosmological expressions [3]. It was pointed out, in effect, that both the translational and the derivation techniques bear world invariance, but no proof of this statement has been given hitherto.

The present paper is mainly aimed at establishing definitely the invariance property brought up above. One of our key procedures includes making use of a new set of coordinate-derivative relations for looking into the world behaviour of a suitably contracted differential configuration that carries a pair of Hermitian connecting objects for the γ-formalism of Infeld and van der Waerden (see, for instance, Ref. [4]). Nevertheless, what crucially appears to control the world characters of all spin-curvature and cosmological expressions taken into consideration herein, amounts to a single mixed-scalar-density prescription for the absolute

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value of the independent component of a γ-formalism metric spinor. In fact, this prescription had been implemented earlier to keep track of the behaviour of a general affine correlation [4]. The compatibility between the classical behaviour of Christoffel connexions and those of the constituents of the world-affinity decomposition just allowed for, gets clearly exhibited out of deducing a system of affine laws.

We will first exhibit in Sect. 2 some of the basic patterns for the class of spacetimes being considered. The affine laws are deduced subsequently in Sect. 3 whereas the derivative relations are shown in Sect. 4. In Sect. 5, we will elaborate upon the specification of the world behaviours at issue. The description of the entire geometric situation will be effectively made up in Sect. 6. Throughout the paper, the primed-unprimed spinor-index notation of Ref. [1] will be adhered to. World components are labelled by lower-case Latin letters. We shall implement the usual convention whereby the effect on any index structure of the actions of the symmetry and antisymmetry operators is indicated by surrounding the indices singled out with round and square brackets, respectively. Unprimed and primed kernel letters are used only to refer to outcomes of invertible world transformations like $x^a \mapsto x'^a(x)$, with the "x" in parentheses meaning functional dependence on some spacetime coordinates x^0, x^1, x^2 and x^3. The partial-derivative operators $\partial/\partial x^a$ and $\partial/\partial x'^a$ will be written as ∂_a and ∂'_a. We will assume at the outset that the signature of an appropriate spacetime metric tensor g_{ab} is $(+---)$. In Sect. 2, we will implicitly allow the sourceless Einstein's equations to carry the cosmological constant. In Sects. 3 through 6, the Jacobian determinant of a world-coordinate transformation will be taken as

$$\delta_W = \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)}.$$

The homogeneous and inhomogeneous parts of some spacetime quantity of arbitrary valence are thus defined so as to enter a configuration of the type

$$T'_{a\ldots b}{}^{p\ldots q}(x') = \mathbf{Hom}[T'_{a\ldots b}{}^{p\ldots q}] + \mathbf{Inh}[T'_{a\ldots b}{}^{p\ldots q}],$$

with the homogeneous part being denoted by

$$\mathbf{Hom}[T'_{a\ldots b}{}^{p\ldots q}] = (\partial'_a x^m)\ldots(\partial'_b x^n)(\partial_r x'^p)\ldots(\partial_s x'^q)T_{m\ldots n}{}^{r\ldots s}(x).$$

These world contributions then obey the outer-product relation

$$\mathbf{Inh}[A'B'] = \mathbf{Hom}[A']\mathbf{Inh}[B'] + \mathbf{Hom}[B']\mathbf{Inh}[A'] + \mathbf{Inh}[A']\mathbf{Inh}[B'],$$

in conformity with the symbolic law

$$C'(x') = \mathbf{Hom}[C'] + \mathbf{Inh}[C'],$$

with the indices of some spacetime quantities A, B and C having been suppressed here for once. When A is regarded as a tensor, we will allow for the "linearity" property

$$\mathbf{Inh}[A'B'] = \mathbf{Hom}[A']\mathbf{Inh}[B'] = A'\mathbf{Inh}[B'].$$

In particular, identifying $(A, B) \equiv (u^a, \eta^a)$, with u^a being a vector, leads to

$$\mathbf{Inh}[u'_a\eta'^a] = g'_{ab}u'^b\mathbf{Inh}[\eta'^a] = u'^a\mathbf{Inh}[g'_{ab}\eta'^b] = \mathbf{Inh}[u'^a\eta'_a].$$

Functional dependences on coordinates will henceforward be omitted. A few further conventions will be explained occasionally.

2 Basic World-Spin Formulae

The world-affine pattern of interest to us is written as

$$\Gamma_{abc} = g_{ab}\Upsilon_c - 2\Upsilon_{(a}g_{b)c}, \tag{1}$$

such that

$$\Gamma_a \doteqdot \Gamma_{ab}{}^b = (-4)\Upsilon_a. \tag{2}$$

It follows immediately from (1) that

$$\partial_a g_{bc} = (-2\Upsilon_a)g_{bc}, \quad \partial_a g^{bc} = 2\Upsilon_a g^{bc}. \tag{3}$$

Consequently, differentiating partially the elementary metric relationship[1]

$$g_{ab} = \sigma_a^{BB'}\sigma_{bBB'}, \tag{4}$$

and manipulating world indices adequately, yields the eigenvalue equations

$$\partial_a\sigma_{bBB'} = (-2\Upsilon_a)\sigma_{bBB'}, \quad \partial_a\sigma^{bBB'} = 2\Upsilon_a\sigma^{bBB'}. \tag{5}$$

One is then led to the constancy property

$$\partial_a\sigma_b^{BB'} = 0, \quad \partial_a\sigma^b_{BB'} = 0. \tag{6}$$

The expression for the γ-formalism spin connexion γ_{aBC} associated to Γ_{abc} is thought of as arising from the simultaneous prescriptions [3, 4]

$$\Gamma_{abc} = \frac{1}{2}\sigma_c^{BB'}\sigma^A_{B'(a}\sigma^{A'}_{b)B}\sigma^h_{AA'}\Gamma_h, \quad \Gamma_{AA'BB'CC'} = \gamma_{AA'B'C'}\gamma_{BC} + \text{c.c.}, \tag{7}$$

with γ_{BC} amounting to one of the metric spinors for the γ-formalism, and the symbol "c.c." standing here as elsewhere for an overall complex conjugate piece. In effect, we have the irreducible decomposition

$$\gamma_{AA'BC} = \theta_{BA'}\gamma_{CA} - i\Phi_{AA'}\gamma_{BC}, \tag{8}$$

with θ_a and Φ_a having normally to be looked upon as world vectors [4]. The quantity θ_a is accordingly expressed as

$$\theta_a = \Upsilon_a - \frac{1}{4}\partial_a \log c, \tag{9}$$

where c is a real world-spin scalar density of world weight -1 and absolute weight $+4$, which bears ∂-constancy in some given world frame [3]. Hence, we can promptly spell out the covariant derivatives

$$\nabla_a w_b = \partial_a w_b - g_{ab}(\Upsilon^c w_c) + 2\Upsilon_{(a}w_{b)}, \tag{10}$$

and

$$\nabla_a u^b = \partial_a u^b - [(\Upsilon_c u^c)\delta_a{}^b + \Upsilon_a u^b - g_{ac}\Upsilon^b u^c], \tag{11}$$

[1] Any σ-symbol will henceforth denote a γ-formalism connecting object.

whose spinor version is supplied by

$$\nabla_{AA'}w_{BB'} = \partial_{AA'}w_{BB'} + \theta_{BA'}w_{AB'} + \theta_{AB'}w_{BA'}, \tag{12}$$

and

$$\nabla_{AA'}u^{BB'} = \partial_{AA'}u^{BB'} - 2\theta_{AA'}u^{BB'} + \theta^{B}_{A'}u^{B'}_{A} + \theta^{B'}_{A}u^{B}_{A'}, \tag{13}$$

with w_a and u^a thus being vectors. The simplest covariant divergences appear as

$$\nabla^a w_a = \Omega^a w_a, \quad \nabla_a u^a = \pi_a u^a, \tag{14}$$

where Ω_a and π_a are defined by

$$\Omega_a \doteqdot \partial_a - 2\Upsilon_a, \quad \pi_a \doteqdot \partial_a - 4\Upsilon_a, \tag{15}$$

and $\partial^a \doteqdot g^{ab}\partial_b$. By appealing to (3), we likewise find the relations

$$\Omega_a B^a - \Omega^a B_a = \partial_a B^a - \partial^a B_a = 2\Upsilon_a B^a, \tag{16}$$

and

$$\Omega_a B^a = \partial^a B_a, \quad \Omega^a B_a = \pi_a B^a, \tag{17}$$

where B^a is some contravariant world vector. Moreover, when acting upon g_{ab} and g^{ab}, the above-defined $\Omega\pi$-operators afford us the statements

$$\Omega_a g_{bc} = (-4\Upsilon_a)g_{bc}, \quad \Omega_a g^{bc} = 0, \tag{18}$$

and

$$\pi_a g_{bc} = (-6\Upsilon_a)g_{bc}, \quad \pi_a g^{bc} = (-2\Upsilon_a)g^{bc}, \tag{19}$$

along with

$$\Omega_a(u^b w_b) = g_{bc}\pi_a(u^b w^c), \quad \pi_a(u^b w_b) = g^{bc}\Omega_a(u_b w_c). \tag{20}$$

The spinor expression for the Riemann tensor of the connexion (1) is written out explicitly as [3]

$$R_{AA'BB'CC'DD'} = \gamma_{A'B'}\Big[\gamma_{CD}(\partial_{(C'(A}\theta_{B)D')} + \theta_{(C'(A}\theta_{B)D')}) + \frac{1}{2}\gamma_{C'D'}\gamma_{C(A}\gamma_{B)D}(\partial^{MM'}\theta_{MM'} - \theta_{MM'}\theta^{MM'})\Big] + \text{c.c.} \tag{21}$$

It turns out that the gravitational spinors of the affinity (8) are expressed as

$$\Xi_{AA'BB'} = \partial_{(A(A'}\theta_{B')B)} + \theta_{(A(A'}\theta_{B')B)}, \tag{22}$$

and[2]

$$\mathrm{X}_{ABCD} = \frac{1}{2}(\partial^{MM'}\theta_{MM'} - \theta_{MM'}\theta^{MM'})\gamma_{A(C}\gamma_{D)B}, \tag{23}$$

[2]We should emphasize that $\gamma_{A(C}\gamma_{D)B} = \gamma_{C(A}\gamma_{B)D}$.

which bring out all the characteristic symmetries [1, 5, 6]. It is evident that the X-spinor of (23) bears a purely cosmological character, that is to say,

$$\Psi_{ABCD} \doteqdot \mathrm{X}_{(ABCD)} \equiv 0, \tag{24}$$

where the Ψ-object is supposedly carried by the spinor representation of the Weyl tensor C_{abcd} of Γ_{abc}, according to the scheme [1, 7]

$$\sigma^a_{AA'}\sigma^b_{BB'}\sigma^c_{CC'}\sigma^d_{DD'}C_{abcd} = \gamma_{A'B'}\gamma_{C'D'}\Psi_{ABCD} + \text{c.c.} \tag{25}$$

The corresponding Ricci spinor is then given by[3]

$$R_{AA'BB'} = \frac{(-3)}{2}(\partial^{MM'}\theta_{MM'} - \theta_{MM'}\theta^{MM'})\gamma_{AB}\gamma_{A'B'} - 2\Xi_{AA'BB'}, \tag{26}$$

which exhibits the cosmological expression [3]

$$\lambda = \frac{(-3)}{2}(\partial^{AA'}\theta_{AA'} - \theta_{AA'}\theta^{AA'}). \tag{27}$$

3 Affine Laws

Obviously, the determinant δ_W defined in Sect. 1 is a homogeneous function of $\partial_a x'^b$ of degree $+4$. Hence, acting on it with the Euler operator [8]

$$\hat{e} = (\partial_a x'^b)\frac{\partial}{\partial(\partial_a x'^b)}, \tag{28}$$

produces the relations

$$\hat{e}\delta_W = (\partial_a x'^b)(\partial'_b x^a)\delta_W = 4\delta_W. \tag{29}$$

Because of the arbitrariness of the coordinates involved in these relations, we can write

$$\frac{\partial\delta_W}{\partial(\partial_a x'^b)} = (\partial'_b x^a)\delta_W \Leftrightarrow (\partial'_b x^a) = \frac{\partial \log \delta_W}{\partial(\partial_a x'^b)}. \tag{30}$$

Thus, resetting $\delta_W = (\Delta_W)^{-1}$ and utilizing the chain rule

$$\partial'_c = \left(\frac{\partial^2 x'^b}{\partial x'^c \partial x^a}\right)\frac{\partial}{\partial(\partial_a x'^b)}, \tag{31}$$

provides the configuration[4]

$$\partial'_a \log \Delta_W = -(\partial'_b x^c)\frac{\partial^2 x'^b}{\partial x'^a \partial x^c} = \frac{\partial^2 x^b}{\partial x^b \partial x'^a}, \tag{32}$$

which amounts to

$$\mathbf{Inh}[\Gamma'_a] = (-4)\mathbf{Inh}[\Upsilon'_a] = \partial'_a \log \Delta_W. \tag{33}$$

[3] Our sign convention for the Ricci tensor is $R_{ab} = R_{ahb}{}^h$.

[4] We stress that partial-derivative operators for different coordinate systems generally do not commute.

Interchanging the roles of the world frames then yields

$$(-4)\mathbf{Inh}[\Upsilon_a] = \partial_a \log(\Delta_W)^{-1} = \frac{\partial^2 x'^b}{\partial x'^b \partial x^a}, \tag{34}$$

whence the implementation of (1) gives rise to the laws

$$\Upsilon'_a = (\partial'_a x^b)\Upsilon_b - \partial'_a \log(\Delta_W)^{1/4}, \tag{35}$$

and

$$\Upsilon_a = (\partial_a x'^b)\Upsilon'_b + \partial_a \log(\Delta_W)^{1/4}. \tag{36}$$

Furthermore, by transvecting (35) and (36) with arbitrary vectors u'^a and u^a, we deduce the property

$$u'^a \partial'_a \log \Delta_W = u^a \partial_a \log \Delta_W, \tag{37}$$

which can also be derived by using Leibniz devices like

$$\frac{\partial^2 x'^b}{\partial x'^b \partial x^a} = -(\partial_a x'^c) \frac{\partial^2 x^b}{\partial x^b \partial x'^c}. \tag{38}$$

It is clear that the procedures leading to the affine law (35) are equivalent to either picking up directly the piece $\mathbf{Inh}[\Gamma'_a]$ or transvecting $\mathbf{Inh}[\Gamma'_{abc}]$ with a suitable metric tensor. Indeed, one can establish the above statement in a subtle way by invoking the symmetric structure

$$\mathbf{Inh}[\Gamma'_{a(bc)}] = g_{ks}(\partial'_{(b} x^k) \frac{\partial^2 x^s}{\partial x'^{c)} \partial x'^a}, \tag{39}$$

along with the computation

$$\begin{aligned} \mathbf{Inh}[\Gamma'_{abc}] &= g_{ks}(\partial'_c x^k) \frac{\partial^2 x^s}{\partial x'^a \partial x'^b} \\ &= \partial'_a g'_{bc} - (\partial'_b x^k)(\partial'_c x^s)(\partial'_a x^h)\partial_h g_{ks} - \mathbf{Inh}[\Gamma'_{acb}] \\ &= g'_{bc}\mathbf{Inh}[(-2)\Upsilon'_a] - \mathbf{Inh}[\Gamma'_{acb}]. \end{aligned} \tag{40}$$

We then obtain the laws

$$\mathbf{Inh}[2\Gamma'_{a(bc)}] = \frac{1}{2} g'_{bc} \frac{\partial^2 x^h}{\partial x^h \partial x'^a} = \mathbf{Inh}[\partial'_a g'_{bc}], \tag{41}$$

which agree with the transvections[5]

$$g'^{bc}\mathbf{Inh}[\Gamma'_{abc}] = (-2)g'^{bc}\mathbf{Inh}[\Gamma'_{bca}] = (-4)\mathbf{Inh}[\Upsilon'_a]. \tag{42}$$

We notice that (39) and (40) produce the pattern

$$2\mathbf{Inh}[\Gamma'_{abc}] = \frac{\partial^2 x^h}{\partial x^h \partial x'^{(a}} g'_{b)c} - \frac{1}{2} g'_{ab} \frac{\partial^2 x^h}{\partial x^h \partial x'^c}, \tag{43}$$

[5] We remark at this point that the affinity (1) satisfies the peculiar relation $\Gamma_{ab}{}^b = (-2)\Gamma^b{}_{ba}$.

which takes us back to the laws (41).

In passing, we point up that the behaviour of Υ'_a may be specified by calling for the rule

$$g^{ks}\partial_k x'^c = g'^{kc}\partial'_k x^s, \tag{44}$$

together the calculation

$$\begin{aligned} g^{ks}\frac{\partial^2 x'^c}{\partial x'^a \partial x^k}\partial_s x'^b &= [\partial'_a(g^{ks}\partial_k x'^c) - (\partial_k x'^c)(\partial'_a x^h)\partial_h g^{ks}]\partial_s x'^b \\ &= g'^{kc}\mathbf{Inh}[\Gamma'_{ak}{}^b] + 2g'^{bc}\mathbf{Inh}[\Upsilon'_a]. \end{aligned} \tag{45}$$

In effect, symmetrizing the configuration (45) over the indices b, c and using the relations (41) gives

$$\begin{aligned} 2g^{ks}\frac{\partial^2 x'^{(b}}{\partial x'^a \partial x^k}\partial_s x'^{c)} &= 2\mathbf{Inh}[\Gamma'_a{}^{(bc)}] + 4g'^{bc}\mathbf{Inh}[\Upsilon'_a] \\ &= 2g'^{bc}\mathbf{Inh}[\Upsilon'_a], \end{aligned} \tag{46}$$

whence we are likewise led to

$$\mathbf{Inh}[\Omega'_a g'^{bc}] = 2g^{ks}\frac{\partial^2 x'^{(b}}{\partial x'^a \partial x^k}\partial_s x'^{c)} + \frac{1}{2}g'^{bc}\frac{\partial^2 x^h}{\partial x^h \partial x'^a} = 0, \tag{47}$$

and

$$\mathbf{Inh}[\pi'_a g'^{bc}] = \frac{1}{2}g'^{bc}\frac{\partial^2 x^h}{\partial x^h \partial x'^a}, \tag{48}$$

in agreement with the equality $\pi_a g^{bc} = 2\Gamma_a{}^{(bc)}$.

The upper-index version of (35) can be achieved by working out the structure which arises out of raising the relevant index. We thus have

$$\begin{aligned} \mathbf{Inh}[4\Upsilon'^a] = \mathbf{Inh}[4g'^{ab}\Upsilon'_b] &= g'^{ab}\partial'_b \log \Delta_W \\ &= -g'^{ab}\frac{\partial^2 x^h}{\partial x^h \partial x'^b} = g^{ks}\frac{\partial^2 x'^b}{\partial x'^b \partial x^k}\partial_s x'^a \\ &= \partial'_b g'^{ab} - \partial_k(g^{ks}\partial_s x'^a) \\ &= \mathbf{Inh}[2\Upsilon'^a] - g^{ks}\frac{\partial^2 x'^a}{\partial x^k \partial x^s}, \end{aligned} \tag{49}$$

whence we can write down the expression

$$\mathbf{Inh}[\Upsilon'^a] = -\frac{1}{2}g^{ks}\frac{\partial^2 x'^a}{\partial x^k \partial x^s}, \tag{50}$$

along with the relation that results from (50) when primed and unprimed kernel letters are interchanged. Actually, there is a much easier, but less elegant, manner of deriving the expression (50). This consists first in calling upon the classical law

$$\mathbf{Inh}[\Gamma'_{bc}{}^a] = (\partial_s x'^a)\frac{\partial^2 x^s}{\partial x'^b \partial x'^c}, \tag{51}$$

and then allowing for the contracted structure $g'^{bc}\Gamma'^{\ a}_{bc}$ together with the prescription

$$(\partial_s x'^a)\frac{\partial^2 x^s}{\partial x'^b \partial x'^c} = -(\partial'_c x^s)\frac{\partial^2 x'^a}{\partial x'^b \partial x^s}. \tag{52}$$

Now, adopting a procedure similar to the one used to derive (37), leads to the property

$$g^{ks}\frac{\partial^2 x'^a}{\partial x^k \partial x^s}u'_a = -g'^{ks}\frac{\partial^2 x^a}{\partial x'^k \partial x'^s}u_a, \tag{53}$$

which evidently coincides with[6]

$$u'_a \mathbf{Inh}[\Upsilon'^a] = -u_a \mathbf{Inh}[\Upsilon^a]. \tag{54}$$

It is of some interest to remark explicitly that the arbitrariness of u'_a and u_a allows one to account for the device

$$g^{ks}\frac{\partial^2 x'^b}{\partial x^k \partial x^s}(\partial'_b x^a) = -g'^{ks}\frac{\partial^2 x^a}{\partial x'^k \partial x'^s}. \tag{55}$$

4 Partial-Derivative Relations

The relationship between the inhomogeneous parts of Υ'_a and Υ'^a as expressed by (33) and (50) can be recovered by utilizing the prescription

$$(\partial_c x'^b)g'_{ab} = (\partial'_a x^b)g_{bc}, \tag{56}$$

and working out the expansion

$$2g^{ks}\frac{\partial^2 x'^b}{\partial x^k \partial x^s}g'_{ab} = g^{ks}\{\partial_k[(\partial_s x'^b)g'_{ab}] - (\partial_s x'^b)\partial_k g'_{ab}\} + (g^{ks} \leftrightarrow g'_{ab}), \tag{57}$$

where $g^{ks} \leftrightarrow g'_{ab}$ amounts to the block which is obtained from the immediately preceding one by interchanging the roles of g^{ks} and g'_{ab}. For the first block of (57), we have the calculation

$$\begin{aligned} &g^{ks}\{\partial_k[(\partial_s x'^b)g'_{ab}] - (\partial_s x'^b)\partial_k g'_{ab}\} \\ &\quad = g^{ks}\{\partial_k[(\partial'_a x^b)g_{bs}] + 2(\partial_s x'^b)(\partial_k x'^r)\Upsilon'_r g'_{ab}\} \\ &\quad = \frac{\partial^2 x^b}{\partial x^b \partial x'^a} + \mathbf{Inh}[2\Upsilon'_a]. \end{aligned} \tag{58}$$

For the second block, we thus have

$$\begin{aligned} &g'_{ab}\{\partial_k[(\partial_s x'^b)g^{ks}] - (\partial_s x'^b)\partial_k g^{ks}\} \\ &\quad = g'_{ab}\{\partial_k[(\partial'_s x^k)g'^{bs}] - 2(\partial_k x'^b)\Upsilon^k\} \\ &\quad = \frac{\partial^2 x^b}{\partial x^b \partial x'^a} + g'_{ab}\mathbf{Inh}[2\Upsilon'^b], \end{aligned} \tag{59}$$

[6] We note that (53) is equivalent to the relation (38).

whence it is legitimate to take account of the statement

$$2g^{ks}\frac{\partial^2 x'^b}{\partial x^k \partial x^s}g'_{ab} = \frac{\partial^2 x^b}{\partial x^b \partial x'^a}. \tag{60}$$

Consequently, transvecting both sides of (60) with g'^{ac} yields

$$2g^{ks}\frac{\partial^2 x'^a}{\partial x^k \partial x^s} = g'^{ac}\frac{\partial^2 x^b}{\partial x^b \partial x'^c}, \tag{61}$$

which produces the desired recovery. Thus, we also have the relation

$$(-2)g'^{ks}\frac{\partial^2 x^a}{\partial x'^k \partial x'^s}u_a = \frac{\partial^2 x^b}{\partial x^b \partial x'^a}u'^a. \tag{62}$$

Let us at this stage introduce the definitions

$$U_{ab} \doteqdot \mathbf{Inh}[\Gamma'_{ab}{}^c u'_c] \Leftrightarrow U_{ab} = \frac{\partial^2 x^c}{\partial x'^a \partial x'^b}u_c = U_{(ab)}, \tag{63}$$

and

$$V_{ab} \doteqdot \mathbf{Inh}[(-4)\Upsilon'_a u'_b] \Leftrightarrow V_{ab} = \frac{\partial^2 x^c}{\partial x^c \partial x'^a}u'_b. \tag{64}$$

If use is made of (1) and (32), we will readily find

$$2U_{ab} = V_{(ab)} - \frac{1}{2}Vg'_{ab}, \tag{65}$$

with

$$V \doteqdot g'^{ks}V_{ks}. \tag{66}$$

Equation (65) amounts to the trace-reversed relation[7]

$$2U_{ab} = \hat{\tau}V_{(ab)}, \tag{67}$$

whence taking the traces of both sides of it leads to

$$(-2)U = V. \tag{68}$$

It follows that, by utilizing the trace-free pattern

$$\hat{s}\Gamma_{ab}{}^c \doteqdot \frac{1}{2}g_{ab}\Upsilon^c - 2\Upsilon_{(a}\delta_{b)}{}^c, \tag{69}$$

and carrying out some computations, we obtain the expressions[8]

$$\mathbf{Inh}[\hat{s}\Gamma'_{ab}{}^c u'_c] = \mathbf{Inh}[\hat{s}\partial'_{(a}u'_{b)}] = \hat{s}U_{ab}, \tag{70}$$

[7]We must use the world metric tensors for the primed frame upon taking the traces of both U_{ab} and V_{ab}. The quantity $\hat{\tau}$ is an operator that reverses the signs of traces while preserving symmetry.

[8]The operator $\hat{s}$ picks out linearly the trace-free parts of any two-world-index configurations, and commutes with $\hat{\tau}$. For example, $\hat{s}U_{ab} = U_{ab} - \frac{1}{4}Ug'_{ab}$. We should observe that $\hat{s}\Gamma_{a(bc)} = 0$.

and

$$(-2)\mathbf{Inh}[\partial'^a u'_a] = V, \tag{71}$$

which reproduce the classical equalities

$$\mathbf{Inh}[\partial'_a u'_b] = \mathbf{Inh}[\Gamma'_{ab}{}^c u'_c] = \mathbf{Inh}[\partial'_{(a} u'_{b)}], \tag{72}$$

in addition to enhancing the world-tensor character of both $\hat{s}\nabla_{(a}u_{b)}$ and $\nabla_a u_b$. Therefore, if we call for the relations

$$\partial^a u^b = g^{ka}\partial_k(g^{bs}u_s) = g^{ka}g^{sb}\partial_k u_s + 2\Upsilon^a u^b, \tag{73}$$

and

$$\Gamma_c{}^{ab}u^c = -[(\Upsilon_c u^c)g^{ab} + 2\Upsilon^{[a}u^{b]}], \tag{74}$$

we will conclude that the corresponding inhomogeneous parts involving u'^a are subject to the prescriptions[9]

$$\mathbf{Inh}[\partial'^a u'^b] = g'^{ka}g'^{sb}\left(U_{ks} - \frac{1}{2}V_{ks}\right), \tag{75}$$

and

$$\mathbf{Inh}[\Gamma'_c{}^{ab}u'^c] = \frac{1}{4}Vg'^{ab} + \frac{1}{2}g'^{ka}g'^{sb}V_{[ks]}, \tag{76}$$

which bring forward the tensor character of $\nabla^a u^b$.

Equations (75) and (76) bring forth an asymmetry property of the affine structure (1), which is related to the fact that $\hat{s}U_{ab}$ and $\hat{s}V_{(ab)}$ do not generally vanish. For establishing this statement in an explicit way, we carry out the calculation

$$\begin{aligned}\mathbf{Inh}[\hat{s}\partial'^{(a}u'^{b)}] &= g'^{ka}g'^{sb}\left(U_{ks} - \frac{1}{2}V_{(ks)}\right) + \frac{1}{4}Vg'^{ab} \\ &= \frac{1}{2}g'^{ka}g'^{sb}(2U_{ks} - \hat{\tau}V_{(ks)}) = 0,\end{aligned} \tag{77}$$

where we have employed (67) and implemented the contracted relation

$$\mathbf{Inh}[\partial'_a u'^a] = \frac{\partial^2 x'^a}{\partial x'^a \partial x^b}u^b = 2U. \tag{78}$$

Thus, the trace-free derivative $\hat{s}\partial'^{(a}u'^{b)}$ is a tensor whilst $\hat{s}\partial'_{(a}u'_{b)}$ is not. Hence, the combination of (71) and (78) gives the expressions

$$\mathbf{Inh}[\partial'_a u'^a] - \mathbf{Inh}[\partial'^a u'_a] = -\frac{1}{2}V, \tag{79}$$

$$\mathbf{Inh}[\Omega'_a u'^a] = U, \tag{80}$$

[9]It should be stressed that $\mathbf{Inh}[\partial'_a u'_b]$ bears symmetry while $\mathbf{Inh}[\partial'^a u'^b]$ does not.

and

$$\mathbf{Inh}[\Omega'^{a}u'_{a}] = \mathbf{Inh}[\pi'_{a}u'^{a}] = U + \frac{1}{2}V = 0, \tag{81}$$

which essentially describe the behaviour of (14). The expression (81) can be obtained in a straightforward way by performing the actual computation

$$\begin{aligned}\mathbf{Inh}[\Omega'^{a}u'_{a}] &= g'^{ab}\left(U_{ab} + \frac{1}{2}V_{ab}\right) \\ &= g^{ks}(\partial_{k}x'^{a})\left(\frac{\partial^{2}x^{b}}{\partial x^{s}\partial x'^{a}}u_{b} + \frac{1}{2}\frac{\partial^{2}x^{b}}{\partial x^{b}\partial x'^{a}}u_{s}\right) \\ &= \frac{1}{2}\frac{\partial^{2}x^{b}}{\partial x^{b}\partial x'^{a}}u'^{a} - g^{ks}\frac{\partial^{2}x'^{a}}{\partial x^{k}\partial x^{s}}u'_{a} = 0,\end{aligned} \tag{82}$$

with the last step of which being due to (60). Additionally, we can right away recover (41) and (46) from the relations

$$\mathbf{Inh}[u'^{c}\partial'_{c}g'_{ab}] = (-2)g'_{k(a}\frac{\partial^{2}x'^{k}}{\partial x'^{b)}\partial x^{s}}u^{s} = \frac{1}{2}Vg'_{ab}, \tag{83}$$

which control the behaviour of (3).

5 Laws for $\partial\sigma$-equations and $\Omega\pi$-structures

The partial-derivative configurations derived in the foregoing section enable us not only to look into the world behaviours of the $\partial\sigma$-eigenvalue equations of Sect. 2, but also to construct typical world-tensor $\Omega\pi$-structures. Towards considering the former part of this statement, we write the expansion

$$\mathbf{Inh}[\partial'_{a}u'_{b}] = u'^{BB'}\mathbf{Inh}[\partial'_{a}\sigma'_{bBB'}] + \sigma'_{bBB'}\mathbf{Inh}[\partial'_{a}(\sigma'^{BB'}_{c}u'^{c})], \tag{84}$$

where

$$u'^{BB'}\mathbf{Inh}[\partial'_{a}\sigma'_{bBB'}] = \frac{1}{2}V_{ab}. \tag{85}$$

Recalling (75) then yields the relation

$$u'^{b}\mathbf{Inh}[\partial'_{a}\sigma'^{BB'}_{b}] = \sigma'^{bBB'}\left\{\left(U_{ab} - \frac{1}{2}V_{ab}\right) - g'_{ak}g'_{bs}\mathbf{Inh}[\partial'^{k}u'^{s}]\right\} = 0. \tag{86}$$

In a quite similar fashion, we may use the equalities (72) to state that the prescriptions

$$u'_{BB'}\mathbf{Inh}[\partial'^{a}\sigma'^{bBB'}] = -\frac{1}{2}g'^{ak}g'^{bs}V_{ks}, \tag{87}$$

and

$$\mathbf{Inh}[\partial'^{a}u'^{b}] = u'_{BB'}\mathbf{Inh}[\partial'^{a}\sigma'^{bBB'}] + \sigma'^{bBB'}\mathbf{Inh}[\partial'^{a}(\sigma'^{c}_{BB'}u'_{c})], \tag{88}$$

imply that[10]

$$u'_b \mathbf{Inh}[\partial'^a \sigma'^b_{BB'}] = 0. \tag{89}$$

When attaining the specification of the behaviours of $\Omega\pi$-derivatives, it may be expedient to allow for the structures

$$\mathbf{Inh}[\Omega'^a u'^b] = g'^{ak} g'^{bs} U_{ks} = \mathbf{Inh}[\Omega'^{(a} u'^{b)}], \tag{90}$$

and

$$\mathbf{Inh}[\Omega'_a u'_b] = U_{ab} + \frac{1}{2} V_{ab}. \tag{91}$$

The law (90) can be derived by employing the operator device

$$g'^{ak} \partial'_k = g^{ks} (\partial_k x'^a) \partial_s, \tag{92}$$

together with (76) and the prescription

$$g'^{ak} \frac{\partial^2 x'^b}{\partial x'^k \partial x^c} u^c = g^{ks} (\partial_k x'^a) \frac{\partial^2 x'^b}{\partial x^s \partial x^c} u^c = -\mathbf{Inh}[\Gamma'^{ab}_c u'^c]. \tag{93}$$

Therefore, after taking suitable traces of the structures (90) and (91), we recover (80) and (81). A more transparent procedure for computing $\mathbf{Inh}[\Omega'^a u'^b]$ involves using (72) and (75). It just rests upon the observation that (16) and (17) allow us to write

$$\Omega^a u^b = g^{ak} g^{bs} \partial_k u_s. \tag{94}$$

We can still point out that the combination of (75) with the relation

$$g_{bc} \Omega_a u^c = \partial_a u_b, \tag{95}$$

produces the whole prescription (90).

Equation (91) provides us with the inhomogeneous part

$$\mathbf{Inh}[\pi'_a u'_b] = \mathbf{Inh}[\Omega'_a u'_b] - 2\mathbf{Inh}[\Upsilon'_a u'_b] = U_{ab} + V_{ab}, \tag{96}$$

which yields the contracted structure

$$2\mathbf{Inh}[\pi'^a u'_a] = V. \tag{97}$$

The behaviour of $\pi^a u^b$ can be specified by either making use of the relation (50) or rewriting (93) in the form

$$g'^{ak} \frac{\partial^2 x'^b}{\partial x'^k \partial x^c} u^c = g'^{ab} \mathbf{Inh}[\Upsilon'_c u'^c] + 2\mathbf{Inh}[\Upsilon'^{[a} u'^{b]}]. \tag{98}$$

In the case of either procedure, we obtain the law

$$\mathbf{Inh}[\pi'^a u'^b] = g'^{ak} g'^{bs} \left(U_{ks} + \frac{1}{2} V_{ks} \right), \tag{99}$$

[10] Writing (85) and (87) presupposes that the properties (6) should remain valid in the primed frame. We will make this point clear in Sect. 6.

whence we can reestablish the invariance of the statements (14) by taking the primed-frame trace of (99).

It is worth working out the symmetrized version of (99). The reason for this concerns the occurrence of a natural trace-free structure on the right-hand side of the symmetrized relation. In effect, by writing

$$\mathbf{Inh}[\pi'^{(a}u'^{b)}] = g'^{c(a}\frac{\partial^2 x'^{b)}}{\partial x'^c \partial x^k}u^k + \frac{\partial^2 x^h}{\partial x^h \partial x'^c}g'^{c(a}u'^{b)}, \tag{100}$$

and carrying out the calculations

$$\begin{aligned}(-1)g'^{c(a}\frac{\partial^2 x'^{b)}}{\partial x'^c \partial x^k}u^k &= (\partial_k x'^{(a})g'^{b)c}\frac{\partial^2 x^k}{\partial x'^c \partial x'^s}u'^s \\ &= \mathbf{Inh}[\Gamma'^{\,(ab)}_{s}u'^s] = \mathbf{Inh}[g'^{ak}g'^{bs}\Gamma'_{c(ks)}u'^c] \\ &= u'^c\mathbf{Inh}\left[g'^{ak}g'^{bs}\left(\frac{1}{2}\partial'_c g'_{ks}\right)\right] = \frac{1}{4}Vg'^{ab},\end{aligned} \tag{101}$$

and

$$\frac{\partial^2 x^h}{\partial x^h \partial x'^c}g'^{c(a}u'^{b)} = \mathbf{Inh}[(-4)\Upsilon'^{(a}u'^{b)}] = g'^{ak}g'^{bs}V_{(ks)}, \tag{102}$$

we get

$$\mathbf{Inh}[\pi'^{(a}u'^{b)}] = g'^{ak}g'^{bs}\hat{s}V_{(ks)}. \tag{103}$$

It is obvious that the structure (103) is consistent with (65). Since tensor symmetries are invariant attributes, we can apply the skew-symmetry operator to the configuration (94) and absorb the property

$$\begin{aligned}\pi^{[a}(u^b w^{c]}) &= (\Omega^{[a}u^b)w^{c]} - u^{[a}(\Omega^b w^{c]}) \\ &= (\Omega^{[[a}u^{b]})w^{c]} - u^{[a}(\Omega^{[b}w^{c]]}),\end{aligned} \tag{104}$$

to obtain the tensor patterns $\Omega^{[a}u^{b]}$ and $\pi^{[a}(u^b w^{c]})$.

6 Concluding Remarks

It has become manifest that the properties (5) and (6) are deeply rooted into the inner structure of the spacetime situation governed by the affinities (1) and (8). Hence, in the unprimed frame, we can surely write

$$\partial^{AA'}\theta_{AA'} = \partial^a\theta_a.$$

However, the primed-frame version of this relation does not really hold because of the genuineness of the simple computation

$$\partial'^{AA'}\theta'_{AA'} = \sigma'^{AA'}_a\partial'^a(\sigma'^b_{AA'}\theta'_b) = \partial'^a\theta'_a + \sigma'^{AA'}_a(\partial'^a\sigma'^b_{AA'})\theta'_b,$$

whence we can say that

$$\partial'^{AA'}\theta'_{AA'} = \partial^{AA'}\theta_{AA'}.$$

In essence, these features rely upon the word-covariant behaviour of θ_a, and thereby establish the invariance of the formulae (21)–(27). They likewise enhance the world invariance of the expansions (12) and (13).

In case the density c of (9) were substituted for a world-invariant spin density, the covariant character of θ_a would be removed from the affine prescriptions. Implementing such a replacement would thus produce the equality

$$\theta'_a = \Upsilon'_a,$$

which, when combined together with the inhomogeneous parts of the expansions

$$\partial'_a \sigma_b'^{AA'} = \gamma^{AB}\gamma^{A'B'}\partial'_a \sigma'_{bBB'} + 2\theta'_a \sigma_b'^{AA'},$$

and

$$\partial'^a \sigma'^b_{AA'} = \gamma_{AB}\gamma_{A'B'}\partial'^a \sigma'^{bBB'} - 2\theta'^a \sigma'^b_{AA'},$$

shows that (6) would be invariant statements. In fact, this substitution procedure brings about the only circumstance under which (86) and (89) could be put into practice, since the utilization of the covariant derivatives

$$\nabla_a \sigma_b^{AA'} = \partial_a \sigma_b^{AA'} - \Gamma_{ab}{}^c \sigma_c^{AA'} + \gamma_{aC}{}^A \sigma_b^{CA'} + \gamma_{aC'}{}^{A'} \sigma_b^{AC'},$$

and

$$\nabla_a \sigma^b_{AA'} = \partial_a \sigma^b_{AA'} + \Gamma_{ac}{}^b \sigma^c_{AA'} - \gamma_{aA}{}^C \sigma^b_{CA'} - \gamma_{aA'}{}^{C'} \sigma^b_{AC'},$$

enables one to verify that (85) and (87) take for granted the applicability of the primed-frame version of (6).

The removal of the covariance of θ_a would entail taking the absolute value of the independent component of γ_{BC} as a world-spin scalar density of world weight $+1$, and the meaning of the traditional eigenvalue equations

$$\partial_a(\gamma^{BC}\gamma^{B'C'}) = (2\theta_a)\gamma^{BC}\gamma^{B'C'},$$

and

$$\partial_a(\gamma_{BC}\gamma_{B'C'}) = (-2\theta_a)\gamma_{BC}\gamma_{B'C'},$$

would be lost. As a consequence, the whole geometric picture of the γ-formalism would have to be eventually reconstructed from the beginning.

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